Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 821 (2009) 228–240

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Majorana fermions and CP-invariance of chiral gauge theories on the lattice

Yuji Igarashi [a,*], Jan M. Pawlowski [b]

[a] *Faculty of Education, Niigata University, Ikarashi, 950-2181, Niigata, Japan*

[b] *Institut für Theoretische Physik, Universität Heidelberg, D-69120 Heidelberg, Germany*

Received 15 April 2009; received in revised form 30 April 2009; accepted 22 June 2009

Available online 27 June 2009

Abstract

The construction of massless Majorana fermions with chiral Yukawa couplings on the lattice is considered. We find topological obstructions tightly linked to those underlying the Nielsen–Ninomiya no-go theorem. In contradistinction to chiral fermions the obstructions originate only from the combination of the Dirac action and the Yukawa term. These findings are used to construct a chirally invariant lattice action. We also show that the path integral of this theory is given by the Pfaffian of the corresponding Dirac operator. As an application of the approach set-up here we construct a CP-invariant lattice action of a chiral gauge theory, based on a lattice adaptation of charge conjugation and parity transformation in the continuum.

PACS: 11.15.-q; 11.15.Ha; 11.30.Er; 11.30.Rd

1. Introduction

Massive neutrinos can be incorporated in the Standard Model with Majorana fermions that become massive via spontaneous symmetry breaking. This mass generation necessitates a chirally symmetric Yukawa term. A lattice approach to the related physics problems has to be based on an appropriate lattice formulation of Majorana fermions in the presence of chiral symmetry [1–5], see also [6–8]. Majorana fermions with chiral symmetry also play a key role for physics beyond the Standard Model, in particular in supersymmetric theories. Realisations of lattice su-

* Corresponding author.
E-mail address: igarashi@ed.niigata-u.ac.jp (Y. Igarashi).

0550-3213/$ – see front matter
doi:10.1016/j.nuclphysb.2009.06.019

persymmetry face additional problems since space–time lattices are generally irreconcilable with supersymmetry, see e.g. [9–15], for reviews see [16–18].

In [8] an approach to chirally coupled Majorana fermions was initiated. Here we extend this work, evaluate the related obstructions and provide a construction of chirally coupled Majorana fermions. This construction may also prove useful for the construction of supersymmetric theories on the lattice. Within a lattice formulation, chiral symmetry becomes non-trivial due to the Nielsen–Ninomiya no-go theorem [19–23], and we expect related obstructions for chirally coupled Majorana fermions. Indeed there appears a certain conflict between the definition of the Majorana fermions and lattice chiral symmetry in the presence of Yukawa couplings. The conflict is closely related to the requirements of locality and that of avoiding species doubling, which are the basic issues of lattice chiral symmetry. It also causes an obstruction in constructing the simplest supersymmetric model, the Wess–Zumino model on a lattice, and in showing CP invariance of chiral gauge theory, see e.g. [5,24,25].

In the first section we recapitulate the continuum formulation of Majorana fermions. Then we discuss the specialties of a lattice formulation, in particular the necessary doubling of degrees of freedom, and the topological obstructions inherited from the demand of chiral invariance. The findings are used to construct Majorana fermions including the proof of the Pfaffian nature of the lattice path integral. Finally we construct a CP-invariant lattice action of a chiral gauge theory based on a lattice extension of charge conjugation and parity transformation.

2. Majorana fermions

Majorana fermions are neutral fermions and hence obey a reality constraint. In four-dimensional Euclidean space–time the charge conjugation operator C has the properties

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T, \qquad C\gamma_5 C^{-1} = \gamma_5^T, \qquad C^\dagger C = \mathbb{1}, \qquad C^T = -C, \tag{1}$$

and the reality constraint for Majorana fermions reads

$$\psi^* = B\psi, \tag{2}$$

where $C = \gamma_5 B$. However, (1) implies that $B^*B = -\mathbb{1}$ and hence we cannot implement the reality constraint (2) as it fails to satisfy the consistency condition $\psi^{**} = \psi$. This problem is circumvented by doubling the degrees of freedom which suffices to implement the reality constraint with

$$\psi^* = \mathcal{B}\psi, \quad \text{with } \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3}$$

and

$$\mathcal{B} = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}.$$

The symplectic structure of $\mathcal{B}$ leads to $\mathcal{B}^*\mathcal{B} = \mathbb{1}$ with $B^*B = -\mathbb{1}$. Thus the reality constraint, $\psi^{**} = \psi$, is satisfied. The corresponding charge conjugation operator is provided by

$$\mathcal{C} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} = \Gamma_5\mathcal{B}, \quad \text{with } \Gamma_5 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix}. \tag{4}$$

The above properties of the symplectic Majorana fermion ψ fix its behaviour under chiral rotations,

$$\psi \to (1 + i\epsilon\Gamma_5)\psi, \tag{5}$$

which introduces a relative minus sign in the chiral rotation of ψ_1 and ψ_2. We are now in the position to construct a chirally invariant Majorana action. We summarise the necessary properties,

$$\mathcal{C} = \Gamma_5 \mathcal{B}, \qquad \mathcal{C}\Gamma_5\mathcal{C}^{-1} = -\Gamma_5^T, \qquad \mathcal{C}^\dagger \mathcal{C} = \mathbb{1}, \qquad \mathcal{C}^T = -\mathcal{C}, \tag{6}$$

and construct the corresponding chirally invariant Majorana action

$$S_D[\psi] = \int d^4x\, \psi^T \mathcal{C}\mathcal{D}\psi = \int d^4x\, \big(\psi_1^T C D \psi_1 + \psi_2^T C D \psi_2\big) \tag{7}$$

with

$$\mathcal{D} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \quad \text{and} \quad (\mathcal{C}\mathcal{D})^T = -\mathcal{C}\mathcal{D}. \tag{8}$$

The action (7) could also be obtained by a Majorana reduction, see e.g. [26,27]. Note that chiral invariance of (7) is trivial, because

$$\psi_i^T C \gamma_5 D \psi_i = 0 \quad (i = 1, 2). \tag{9}$$

We also remark that skew symmetry of $\mathcal{D}$ is not required but only the skew-symmetric part of $\mathcal{D}$ contributes to the action S. The definitions (8) imply

$$(CD)^T = -CD \quad \text{and} \quad D^* = BDB^{-1}, \tag{10}$$

for the Dirac operator D. The combined properties (10) hold for the standard chiral Dirac operator with

$$\gamma_5 D + D\gamma_5 = 0, \tag{11}$$

such as $D = \gamma_\mu \partial_\mu$, for which (10) can be deduced from (1). The action (7) is chirally invariant under a chiral transformation with (5) if

$$\Gamma_5 \mathcal{D} - \mathcal{D}\Gamma_5 = 0, \tag{12}$$

which is valid for $\mathcal{D}$ with the standard Dirac operator satisfying (11). Finally we remark that the action (7) is real, as follows from (10). It is instructive to make this reality explicit by rewriting the action (7) with the help of the above relations,

$$S_D[\psi] = \int d^4x\, \big(\psi_2^\dagger \gamma_5 D \psi_1 + \psi_1^\dagger \gamma_5 D^\dagger \psi_2\big). \tag{13}$$

Note that (13) is even real for unconstrained Dirac fermions ψ_1, ψ_2 in contrast to (7).

Eqs. (7), (13) provide the action of a Euclidean Majorana fermion. For the chirally coupled Yukawa theory we are also interested in Weyl fermions which we can construct from the components of the Majorana spinor (ψ_1, ψ_2). To that end let us introduce chiral projection operators related to Γ_5 in (5),

$$\mathcal{P} = \frac{1}{2}(1 + \Gamma_5) = \begin{pmatrix} P & 0 \\ 0 & 1 - P \end{pmatrix}, \tag{14}$$

with

$$P = \frac{1}{2}(1 + \gamma_5).$$

The chiral projection operators $\mathcal{P}$, $(1 - \mathcal{P})$ allow us to project on right-handed and left-handed spinors. For a given pair of ψ_1 and ψ_2, we can construct a Weyl action taking its off-diagonal combination:

$$S_W[\psi]=\int d^4x\,\psi^T\mathcal{C}\begin{pmatrix}0&0\\1&0\end{pmatrix}\mathcal{D}\mathcal{P}\psi=\int d^4x\,\psi_2^T C D P\psi_1. \tag{15}$$

We shall use this type of action to discuss CP invariance of a chiral gauge theory on the lattice.

For the construction of a chirally invariant Yukawa term we have to couple chiral projections of the Majorana fermions to the scalar field,

$$\begin{aligned}S_Y[\psi,\phi]&=g\int d^4x\,\big(\psi^T\mathcal{C}\mathcal{P}\phi^\dagger\psi+\psi^T\mathcal{C}(1-\mathcal{P})\phi\psi\big)\\&=g\int d^4x\,\big(\psi_1^T CP\varphi\psi_1+\psi_1^T C(1-P)\varphi^\dagger\psi_1\\&\quad+\psi_2^T CP\varphi^\dagger\psi_2+\psi_2^T C(1-P)\varphi\psi_2\big).\end{aligned} \tag{16}$$

In (16) φ is a complex scalar, and ϕ is defined by

$$\phi=\begin{pmatrix}0&\varphi\\\varphi&0\end{pmatrix},\quad \text{with } \phi\to(1-2i\epsilon)\phi. \tag{17}$$

Note that the scalar field ϕ is off-diagonal and hence does not commute with the projection operators, we have e.g. $\mathcal{P}\phi^\dagger=\phi^\dagger(1-\mathcal{P})$. The action $S_D[\psi]+S_Y[\psi,\phi]$ is invariant under the transformation (5) of the fermions and that in (17) for the scalar field ϕ related to $\varphi\to(1-2i\epsilon)\varphi$. This concludes our brief summary of chirally coupled Majorana fermions in the continuum. Due to chiral symmetry, and in particular the use of chiral projections in (16) we expect obstructions for putting the above theory on the lattice.

3. Topological obstructions on the lattice

Chiral symmetry on the lattice differs from that in the continuum as consistent chiral transformations necessarily depend on the Dirac operator. Hence we first discuss the properties of the lattice version of the free Dirac action (7)

$$S_D[\psi]=\sum_{x,y\in\Lambda}\Psi^T(x)\mathcal{C}\mathcal{D}(x-y)\Psi(y), \tag{18}$$

with the lattice Dirac operator $D(x-y)$ used in the definition of $\mathcal{D}$ as defined in (8), and $\Psi=(\Psi_1,\Psi_2)$ obeys the symplectic reality constraint (3). Assume for the moment that $D(x-y)$ is of Ginsparg–Wilson type [1] with

$$\gamma_5 D+D\gamma_5=aD\gamma_5 D, \tag{19}$$

related to a vector-symmetric blocking procedure. Then the chiral transformation

$$\Psi\to\left[1+i\epsilon\Gamma_5\left(1-\frac{1}{2}a\begin{pmatrix}0&1\\1&0\end{pmatrix}\mathcal{D}\right)\right]\Psi, \tag{20}$$

is an invariance of (18). However, smooth chiral projections P and $\mathcal{P}$ cannot be constructed, which is reflected in the fact that the transformation (20) vanishes at the doublers. This is a consequence of the well-known Nielsen–Ninomiya no-go theorem [19–23], which provides obstructions for putting chiral fermions on the lattice. Ginsparg–Wilson fermions [1] circumvent the no-go theorem with a modified chiral symmetry (19), which can be reformulated as

$$\gamma_5 D + D\hat{\gamma}_5 = 0, \quad \text{with } \hat{\gamma}_5 = \gamma_5(1 - aD), \tag{21}$$

and chiral projections

$$P = \frac{1}{2}(1+\gamma_5), \qquad \hat{P} = \frac{1}{2}(1+\hat{\gamma}_5). \tag{22}$$

The general case [15,23], going beyond Ginsparg–Wilson fermions, including e.g. [28], only resorts to general chiral projections P, $\hat{P}$, which are compatible:

$$(1 - P)D = D\hat{P}. \tag{23}$$

It has been shown in [23] that projection operators P, $\hat{P}$ carry a winding number that is related to the total chirality χ of the system at hand,

$$\chi = n[\hat{P}] - n[1 - P], \tag{24}$$

with

$$n[P] \equiv \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \int\limits_{T^4} \mathrm{tr}\, P\,(dP)^4 \in \mathbf{Z},$$

if $\hat{P}\Psi = \Psi$ in the action. Eq. (24) also entails that for odd chirality χ, $\hat{P}\Psi$ and $P\Psi$ live in topologically different spaces, and hence have to be different. In the present case the total chirality χ is even due to the symplectic construction. The continuum Yukawa action, however, contains projection operators $\mathcal{P}$, $1-\mathcal{P}$ with P, $1-P$ on chiral sub-spaces with $\mathcal{P}\Psi \neq \Psi$, that is on fermionic sub-systems with odd chirality. Thus we have to worry about the use of projection operators in the Yukawa action S_Y.

The first question that arises in this context is whether the lattice Yukawa action can be constructed such that it remains invariant under the chiral transformations (20), and tends toward the continuum action. This would require the existence of a smooth operator $\tilde{P}$ which reduces $\tilde{P} \to 1 - P$ in the continuum limit, and ensures invariance of the Yukawa term under the combined transformation (17) and (20). Furthermore, there is no transformation of the scalar field ϕ that could absorb a momentum-dependent transformation of $\tilde{P}\Psi$. Hence, a necessary condition for the invariance of the Yukawa term is a transformation of $\tilde{P}\Psi$ that is independent of the Dirac operator D,

$$\tilde{P}\Psi \to \pm\epsilon M \tilde{P}\Psi, \tag{25}$$

with constant matrix M. As $\gamma_5(1 - \frac{a}{2}D)$ is not normalised and even vanishes at the doublers such an operator $\tilde{P}$ cannot exist, even if one relaxes the projection property $\tilde{P}^2 = \tilde{P}$.

In turn it is required that the chiral transformation must be compatible with the projection operators used in the Yukawa term. This already excludes (20). Without loss of generality we can restrict ourselves to the chiral transformation

$$\Psi \to (1 + i\epsilon\hat{\Gamma}_5)\Psi, \quad \text{with } \hat{\Gamma}_5 = \begin{pmatrix} \hat{\gamma}_5 & 0 \\ 0 & -\hat{\gamma}_5 \end{pmatrix}, \tag{26}$$

and hence

$$\Psi^T\hat{\mathcal{C}} \to \Psi^T\hat{\mathcal{C}}\left[\hat{\mathcal{C}}^{-1}\left(1 + i\epsilon\hat{\Gamma}_5^T\right)\hat{\mathcal{C}}\right],$$

where $\hat{\mathcal{C}}$ is a lattice generalisation of $\mathcal{C}$. Then, chiral invariance of the action S in (18) leads to the constraint

$$\hat{\mathcal{C}}^{-1}\hat{\Gamma}_5^T\hat{\mathcal{C}} = -\Gamma_5, \quad \text{with } \Gamma_5\mathcal{D} = \mathcal{D}\hat{\Gamma}_5. \tag{27}$$

We conclude that invariance of the lattice action (18) under the chiral transformations (26) would require

$$\hat{\gamma}_5^T = \hat{C}\gamma_5\hat{C}^{-1}, \tag{28}$$

which maps $\hat{\gamma}_5$ carrying the winding number $n[\hat{P}]$ to γ_5 carrying the winding number $n[P]$. Note that using different γ_5's in the definition of Γ_5 still leads to the same conclusion (28). In order to elucidate this obstruction we use Ginsparg–Wilson fermions as an example. There the relation (28) reads

$$\left(1 - aD^T\right)\gamma_5^T = \hat{C}\gamma_5\hat{C}^{-1}, \tag{29}$$

with a possible solution

$$\hat{C} = C\left(1 - \frac{1}{2}aD\right). \tag{30}$$

The $\hat{C}$ in (30) has zeros at the doublers, and the relative winding number is carried by these zeros. Inserting a lattice $\hat{C}$ in (30) into the action (18) we encounter zeros or singularities of the operator $\hat{C}^{-1}D$ at the positions of the doublers. This brings back the doubling problem. Consequently, we have to use independent Majorana fields ψ, Ψ with different chiral transformation properties for the construction of Majorana actions. At the same time, our discussion given here also suggests the possibility of defining a lattice generalisation of charge conjugation as a discrete transformation between these topologically different spinors. We shall discuss first the construction of Majorana actions, and then how to define charge conjugation in connection with CP invariance of chiral gauge theories.

4. Construction of Majorana actions

Now we are in the position to construct chirally coupled Majorana fermions on the lattice. In line with the arguments of the last section we introduce two sets of symplectic Majorana fermion, ψ and Ψ. Then chiral invariance is easily arranged for with appropriate, different, chiral transformations for ψ and Ψ respectively. Furthermore, we have to ensure that our path integral results in Pfaffians of the Dirac operator which signals Majorana fermions. The corresponding lattice action reads

$$S_D[\psi, \Psi] = \sum_{x,y\in\Lambda} \psi^T(x)\mathcal{C}\mathcal{D}(x-y)\Psi(y), \tag{31}$$

with the Yukawa term

$$\begin{aligned} S_Y[\psi, \Psi, \phi] &= g\sum_{x,y\in\Lambda}\left(\psi^T\mathcal{C}\mathcal{P}\phi^\dagger(1-\hat{\mathcal{P}})\Psi + \psi^T\mathcal{C}(1-\mathcal{P})\phi\hat{\mathcal{P}}\Psi\right) \\ &= g\sum_{x,y\in\Lambda}\left(\psi_1^T CP\varphi\hat{P}\Psi_1 + \psi_1^T C(1-P)\varphi^*(1-\hat{P})\Psi_1\right. \\ &\qquad \left. + \psi_2^T CP\varphi^*\hat{P}\Psi_2 + \psi_2^T C(1-P)\varphi(1-\hat{P})\Psi_2\right), \end{aligned} \tag{32}$$

where

$$\mathcal{P} = (1 + \Gamma_5)/2, \qquad P = (1 + \gamma_5)/2,$$
$$\hat{\mathcal{P}} = (1 + \hat{\Gamma}_5)/2, \qquad \hat{P} = (1 + \hat{\gamma}_5)/2. \tag{33}$$

Note that the scalar field φ does not commute with the projection operators P, $\hat{P}$, as they depend in general on the Dirac operator. Hence, in (32) the combinations such as $P\varphi\hat{P}$ cannot be reduced to $P\hat{P}\varphi$. This is in contrast to the standard definition, see e.g. [29]. We also remark that the construction given here differs from that in [4,30]. There additional Majorana fermions were introduced as static auxiliary fields, and are used to construct a chirally invariant Yukawa term, see also [31–33].

The action $S_D + S_Y$ is invariant under the chiral transformations

$$\Psi \to (1 + i\epsilon\hat{\Gamma}_5)\Psi, \qquad \psi \to (1 + i\epsilon\Gamma_5)\psi, \qquad \phi \to (1 - 2i\epsilon)\phi. \tag{34}$$

The action $S + S_Y$ reduces to the continuum action in the continuum limit, but with a doubling of the field content. This doubling can be removed by appropriate prefactors in the action, or by simply taking roots of the generating functional Z. However, it is left to prove the Pfaffian nature of the path integral. Since we have doubled the degrees of freedom we could have constructed a Dirac fermion out of two Majorana fermions. For the proof it is sufficient to concentrate on the path integral of the free Majorana action [34–36] including a mass term for dealing with the zero modes:

$$Z = \int \prod_x d\psi_1\, d\psi_1^*\, d\Psi_1\, d\Psi_1^* \exp -\big(S_D[\psi,\Psi] + S_m[\psi,\Psi]\big), \tag{35}$$

where the Majorana action (31) couples ψ_1 to Ψ_1 and ψ_2 to Ψ_2 respectively,

$$\begin{aligned} S_D[\psi,\Psi] &= \sum_{x,y\in\Lambda} \psi^T(x)\mathcal{C}\mathcal{D}(x-y)\Psi(y) \\ &= \sum_{x,y\in\Lambda} \big(\psi_1^T(x)CD(x-y)\Psi_1(y) + \psi_2^T(x)CD(x-y)\Psi_2(y)\big). \end{aligned} \tag{36}$$

This also applies to the regularising mass term

$$\begin{aligned} S_m[\psi,\Psi] &= im \sum_{x\in\Lambda} \psi^T(x)\mathcal{C}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\Gamma_5\Psi)(x) \\ &= im \sum_x \big(\psi_1^T(x)C\gamma_5\Psi_1(x) - \psi_2^T(x)C\gamma_5\Psi_2(x)\big). \end{aligned} \tag{37}$$

The Pfaffian is computed in terms of eigenvalues of the Dirac operator $\mathcal{C}\mathcal{D}$. Hence we proceed by expanding the fermionic fields ψ, Ψ, and consequently the action (36), (37) in terms of eigenfunctions of $\mathcal{C}\mathcal{D}$. The operator $\mathcal{C}\mathcal{D}$ can be constructed from the hermitian operator $H = \gamma_5 D$ and its corresponding eigenfunctions φ_n by

$$\sum_y H(x-y)\varphi_n(y) = \lambda_n\varphi_n(x), \tag{38}$$

with

$$(\varphi_n, \varphi_m) = \sum_x \varphi_n^\dagger(x)\varphi_m(x) = \delta_{n,m}.$$

It straightforwardly follows that there is a double degeneracy in these equations: from the eigenfunctions φ we can construct linearly independent eigenfunctions $\phi_n = \gamma_5 C^{-1}\varphi_n^*$ with eigenvalues λ_n, that is $H(x)\phi_n(x) = \lambda_n\phi_n(x)$, and $(\phi_n, \varphi_m) = 0$. Therefore, the fields Ψ_1 and ψ_1 can be expanded as

$$\Psi_1(x) = \sum_n \big[\varphi_n(x)c_n + \phi_n(x)b_n\big],$$
$$\psi_1(x) = \sum_n \big[\varphi_n(x)c'_n + \phi_n(x)b'_n\big]. \tag{39}$$

Inserting this expansion into the action (36), (37) we are led to

$$S_D[\psi, \Psi] + S_m[\psi, \Psi] = -\sum_n \big[(\lambda_n + im)\big(b'_n c_n + b_n c'_n\big) + \text{c.c.}\big]. \tag{40}$$

It is left to rewrite the path integral measure as a measure of integrations over the coefficients c_n, b_n and c'_n, b'_n. This simple but technical derivation is deferred to Appendix A. With the abbreviation $N_{\max} = 1/2\,\mathrm{Tr}\,\mathbb{1}$ the result is

$$\prod_x d\psi_1\, d\psi_1^*\, d\Psi_1\, d\Psi_1^* = \left(\prod_{n=1}^{N_{\max}} dc_n \prod_{n=1}^{N_{\max}} db_n \prod_{n=1}^{N_{\max}} dc'_n \prod_{n=1}^{N_{\max}} db'_n\right)$$
$$\times\left(\prod_{n=1}^{N_{\max}} dc_n \prod_{n=1}^{N_{\max}} db_n \prod_{n=1}^{N_{\max}} dc'_n \prod_{n=1}^{N_{\max}} db'_n\right)^*. \tag{41}$$

Using these results we obtain

$$Z = \prod_{\lambda_n} \big(\lambda_n^2 + m^2\big)^2. \tag{42}$$

Except for the zero modes and the biggest eigenvalues $\lambda_{\max} = 2/a$, all eigenvalues come in pairs $\pm\lambda_n$. More explicitly, the related eigenfunction φ_n^- with $H\varphi_n^- = -\lambda_n\varphi_n^-$ is provided by

$$\varphi_n^- = \frac{1}{\sqrt{1 - a^2\lambda_n^2/4}}\gamma_5(1 - a/2D)\varphi_n. \tag{43}$$

Note that the above operator acting on φ_n provides a normalised γ_5. It cannot be extended to all eigenfunctions because of the topological obstructions [23]. Such a construction precisely fails at the largest eigenvalues $\pm\lambda_{\max}$, as can be seen also from (43).

Let n_+ and n_- be a number of zero modes with positive and negative chiralities, and N_+ and N_- be a number of the eigenfunctions with eigenvalues $\pm\lambda_{\max}$. Then we conclude

$$Z = m^{2(n_+ + n_-)}\left(\frac{4}{a^2} + m^2\right)^{N_+ + N_-} \prod_{0<\lambda_n \neq 2/a} \big(\lambda^2 + m^2\big)^4, \tag{44}$$

and for the massless limit

$$Z = \left(\frac{4}{a^2}\right)^{N_+ + N_-} \prod_{0<\lambda_n \neq 2/a} \lambda_n^8. \tag{45}$$

In other words, the partition function (35) can be expressed in terms of Pfaffian:

$$Z = \mathrm{PF}(CD)^2\, \mathrm{PF}\big(C^* D^*\big)^2, \tag{46}$$

as is mandatory for our construction. This concludes the derivation of a lattice action for Majorana fermions.

5. CP invariance in chiral gauge theories

It has been widely recognised that there is an obstruction in showing CP invariance of chiral gauge theories, see e.g. [24,25]. Consider the standard lattice action of a chiral gauge theory

$$S[\psi,\bar\psi,U]=\sum_{x,y\in\Lambda}\bar\psi(x)\left(\frac{1-\gamma_5}{2}\right)D(U)(x-y)\psi(y), \tag{47}$$

with $\gamma_5 D(U)+D(U)\hat\gamma_5(U)=0$, and hence the chiral action in (47) has the property $\bar\psi(1-P)D\psi=\bar\psi D\hat P\psi$ with the chiral projection operators P, $\hat P$ defined (33). The Ginsparg–Wilson Dirac operator, $D(U)$, now depends on the link variable for gauge field U, as does $\hat\gamma_5(U)$,

$$\hat\gamma_5(U)=\gamma_5\bigl(1-aD(U)\bigr). \tag{48}$$

We infer from (48) that also the chiral projection operator $\hat P$ now depends on the link variable via $\hat\gamma_5$. We also remark that in [37,38] an alternative approach with gauge field independent projection operators has been set-up. This approach again utilises the doubling of fields mandatory due to the no-go theorem.

Now we proceed to the question of a CP-invariant lattice action. It follows from the above discussion that the no-go theorem [23] prevents the construction of a chiral lattice action that is invariant under the standard CP-transformation in the continuum. Note however that the no-go theorem is not an obstruction for the construction of CP-invariance on the lattice as CP is a discrete transformation. Still we can infer that a lattice generalisation of CP necessarily depends on the link variable. In the following we shall consider non-trivial lattice generalisations of either charge conjugation or parity transformation.

Here we put forward a construction where we keep the standard parity transformation and provide a lattice modification of charge conjugation. We emphasise again that this construction is not unique, we can modify both, parity and charge conjugation. This is exemplified in Appendix B for modified lattice parity and standard charge conjugation. The standard parity transformations relevant in the present construction are given by

$$\begin{aligned}\psi(x)&\to\psi^P(x)=P^{-1}\psi(x_P),\\ \bar\psi(x)&\to\bar\psi^P(x)=\bar\psi(x_P)P,\end{aligned} \tag{49}$$

where P denotes the standard parity transformation matrix, and $x_P=(-x_1,-x_2,-x_3,x_4)$. The properties of the link variable U_μ under parity transformations are summarised in

$$U_\mu(x)\to U^P_\mu(x)=\begin{cases}U_i^\dagger(x_P-a\hat i),\\ U_4(x_P),\end{cases} \tag{50}$$

for $i=1,2,3$. For the Dirac operator, we find

$$PD\bigl(U^P\bigr)P^{-1}(x,y)=D(U)(x_P,y_P). \tag{51}$$

We proceed with the observation that the natural choice for the charge conjugation properties of the link variable U and the Dirac operator D are the continuum properties. In the Dirac operator we have not included the chiral projections and hence both, D and U, are insensitive to the topological obstructions related to the chiral properties of the theory. For the link variable U charge conjugation reads

$$U_\mu(x) \to U_\mu^C(x) = \left(U_\mu^\dagger\right)^T(x). \tag{52}$$

Eq. (52) also enters the charge conjugation for the Dirac operator, which is given by

$$CD\left(U^C\right)C^{-1} = \left(D(U)\right)^T. \tag{53}$$

In the case of the fermions $\psi, \bar{\psi}$ charge conjugation defines a mapping between spinors with given chirality defined in terms of the chiral projection operators P, $\hat{P}$. Therefore, we need to define charge conjugation including chiral projections as

$$\begin{aligned} &\bar{\psi}(x)\frac{1\pm\gamma_5}{2} \to \left(\psi^T C\frac{1\pm\tilde{\gamma}_5(U^C)}{2}\right)(x), \\ &\left(\frac{1\pm\hat{\gamma}_5(U)}{2}\psi\right)(x) \to -\frac{1\pm\gamma_5}{2}C^{-1}\bar{\psi}^T(x), \end{aligned} \tag{54}$$

where

$$\tilde{\gamma}_5(U) = \left(1 - aD(U)\right)\gamma_5. \tag{55}$$

As indicated before, the transformation (54) provides a (discrete) mapping between topologically different spinor spaces. Note that our definition of charge conjugation (54) reduces to

$$\psi \to -C^{-1}\bar{\psi}^T, \qquad \bar{\psi} \to \psi^T C, \tag{56}$$

in the continuum limit. Collecting the above results, it is straightforward to show that the action (47) is CP-invariant. Note that the functional measure can be constructed as put forward in [5], using the chiral projection operators P, $\hat{P}$. It is invariant under the CP transformation with (54) and (49).

Our lattice extension of the charge conjugation also applies to the symplectic Majorana fermions:

$$\begin{aligned} &\psi_a^T(x)C\frac{1\pm\gamma_5}{2} \to \varepsilon_{ab}\left(\Psi_b^T C\frac{1\pm\tilde{\gamma}_5(U^C)}{2}\right)(x), \\ &\left(\frac{1\pm\hat{\gamma}_5(U)}{2}\Psi_a\right)(x) \to \varepsilon_{ab}\frac{1\pm\gamma_5}{2}\psi_b(x), \end{aligned} \tag{57}$$

where $a, b = 1$ or 2. Parity transformation reads

$$\begin{aligned} &\Psi_1(x) \to \Psi_1^P(x) = \eta P^{-1}\Psi_1(x_P), \\ &\Psi_2(x) \to \Psi_2^P(x) = -\eta P^{-1}\Psi_2(x_P), \\ &\psi_1^T(x)C \to \left(\psi_1^T(x)C\right)^P(x) = \eta\psi_1^T(x_P)CP, \\ &\psi_2^T(x)C \to \left(\psi_2^T(x)C\right)^P(x) = -\eta\psi_2^T(x_P)CP, \end{aligned} \tag{58}$$

where $\eta = \pm 1$. Then, we can show that a chiral gauge theory described by the Majorana–Weyl action

$$S_{\mathrm{MW}}[\psi, \Psi] = \sum_{x,y\in\Lambda}\psi_2^T(x)C\left(\frac{1-\gamma_5}{2}\right)D(U)(x-y)\Psi_1(y) \tag{59}$$

is CP-invariant.

6. Conclusion

We have shown that the construction of a theory with chirally coupled Majorana fermions on the lattice has to deal with the usual topological obstructions well-known from the construction of chiral fermions, even though the total chirality is even. The obstruction is related to the use of chiral projection operators in the Yukawa term. This problem is resolved by doubling the degrees of freedom, and the Pfaffian nature of the path integral is proven.

We have also shown that the difficulty of constructing CP-invariant chiral gauge theories arises from the same topological obstruction. In order to derive a CP-invariant action for a chiral gauge theory, we introduce a lattice generalisation of the CP transformation. This was done by modifying either lattice parity or charge conjugation or both. The construction also applies to Majorana–Weyl fermions. It remains to be seen how amiable these constructions are towards numerical implementation, see e.g. [39–45]. To that end one also should further explore the generality of the approach presented here in order to have maximal flexibility within the numerical implementation. Another interesting direction is the extension of the construction presented here to supersymmetric theories. In the continuum theory, symplectic Majorana fermions can be used to discuss extended supersymmetric theories. It is therefore challenging to construct such theories on the lattice.

Acknowledgements

Y.I. would like to thank the Institute of Theoretical Physics in Heidelberg for hospitality. We are grateful to F. Bruckmann and H. Suzuki for various useful discussions.

Appendix A. Path integral measure

Here we rewrite the path integral measure over ψ, Ψ in terms of the expansion coefficients c_n, b_n and c'_n, b'_n. We start with

$$\prod_x d\psi_1 = \det V^{-1} \prod_{n=1}^{N_{\max}} dc_n \prod_{n=1}^{N_{\max}} db_n, \tag{A.1}$$

where $N_{\max} = 1/2\,\mathrm{Tr}\,\mathbb{1}$, and the matrix V is defined by

$$V_{x,n} = \left(\varphi_n(x), \gamma_5 C^{-1}\varphi_n^*(x)\right). \tag{A.2}$$

The matrix V has the properties

$$\sum_{x,l} V^T_{m,x} B^{-1} V_{x,l} \begin{pmatrix} 0 & -\delta_{l,n} \\ \delta_{l,n} & 0 \end{pmatrix} = \begin{pmatrix} \delta_{m,n} & 0 \\ 0 & \delta_{m,n} \end{pmatrix}, \tag{A.3}$$

and

$$(\det V)^{-2} = (-)^{N^2_{\max}+N^2_{\max}} \det\left(\gamma_5 C^{-1}\right). \tag{A.4}$$

With help of these relations we conclude that (41) holds.

Appendix B. Modified lattice parity

Let us discuss a lattice generalisation of parity transformation, realising charge conjugation in the standard way. For the action (47), we adopt the charge conjugation (56). The parity transformation is required to give a mapping between topologically different spinor states with opposite chiralities. Therefore, we need to introduce chiral projection operators explicitly for a lattice generalisation of the parity transformations,

$$\begin{aligned}
&\frac{1\pm\gamma_5}{2}\psi(x) \to P^{-1}\left(\frac{1\mp\hat{\gamma}_5(U)}{2}\psi\right)(x_P), \\
&\left(\bar{\psi}\frac{1\pm\tilde{\gamma}_5(U)}{2}\right)(x) \to \bar{\psi}(x_P)\frac{1\mp\gamma_5}{2}P.
\end{aligned} \tag{B.1}$$

The action (47) is shown to be invariant under combined operations (56) and (B.1).

Let us consider the Majorana–Weyl action (59). The parity transformations should flip the chirality which is defined with different projection operators for ψ and Ψ. It reads for the fermions ψ,

$$\begin{aligned}
&\frac{1\pm\gamma_5}{2}\psi_1(x) \to \eta P^{-1}\left(\frac{1\mp\hat{\gamma}_5(U)}{2}\Psi_1\right)(x_P), \\
&\frac{1\pm\gamma_5}{2}\psi_2(x) \to -\eta P^{-1}\left(\frac{1\mp\hat{\gamma}_5(U)}{2}\Psi_2\right)(x_P).
\end{aligned} \tag{B.2}$$

For the fermion Ψ we define

$$\begin{aligned}
&\left(\frac{1\pm\hat{\gamma}_5(U)}{2}\Psi_1\right)(x) \to \eta P^{-1}\frac{1\mp\gamma_5}{2}\psi_1(x_P), \\
&\left(\frac{1\pm\hat{\gamma}_5(U)}{2}\Psi_2\right)(x) \to -\eta P^{-1}\frac{1\mp\gamma_5}{2}\psi_2(x_P).
\end{aligned} \tag{B.3}$$

Collecting the above results and using (56), it is straightforward to show that Majorana–Weyl action (59) is CP-invariant.

References

[1] P.H. Ginsparg, K.G. Wilson, Phys. Rev. D 25 (1982) 2649.
[2] H. Neuberger, Phys. Lett. B 417 (1998) 141;
H. Neuberger, Phys. Rev. D 57 (1998) 5417;
H. Neuberger, Phys. Lett. B 427 (1998) 353.
[3] P. Hasenfratz, V. Laliena, F. Niedermayer, Phys. Lett. B 427 (1998) 125.
[4] M. Lüscher, Phys. Lett. B 428 (1998) 342.
[5] M. Lüscher, Nucl. Phys. B 549 (1999) 295.
[6] K. Fujikawa, M. Ishibashi, Phys. Lett. B 528 (2002) 295.
[7] H. Suzuki, Prog. Theor. Phys. 112 (2004) 855.
[8] J.M. Pawlowski, Y. Igarashi, PoS LAT2007 (2007) 275.
[9] P.H. Dondi, H. Nicolai, Nuovo Cimento A 41 (1977) 1.
[10] G. Curci, G. Veneziano, Nucl. Phys. B 292 (1987) 555.
[11] K. Fujikawa, Nucl. Phys. B 636 (2002) 80;
K. Fujikawa, Phys. Rev. D 66 (2002) 074510.
[12] F. Sugino, JHEP 0401 (2004) 015.
[13] A. D'Adda, I. Kanamori, N. Kawamoto, K. Nagata, PoS LAT2007 (2007) 271.
[14] M. Kato, M. Sakamoto, H. So, JHEP 0805 (2008) 057.

[15] G. Bergner, F. Bruckmann, J.M. Pawlowski, arXiv:0807.1110 [hep-lat].
[16] A. Feo, Nucl. Phys. B (Proc. Suppl.) 119 (2003) 198.
[17] D.B. Kaplan, Nucl. Phys. B (Proc. Suppl.) 129 (2004) 109;
D.B. Kaplan, Eur. Phys. J. ST 152 (2007) 89.
[18] J. Giedt, Int. J. Mod. Phys. A 21 (2006) 3039–3094.
[19] L.H. Karsten, J. Smit, Nucl. Phys. B 183 (1981) 103.
[20] H.B. Nielsen, M. Ninomiya, Nucl. Phys. B 185 (1981) 20;
H.B. Nielsen, M. Ninomiya, Nucl. Phys. B 195 (1981) 541, Erratum;
H.B. Nielsen, M. Ninomiya, Nucl. Phys. B 193 (1981) 173.
[21] L.H. Karsten, Phys. Lett. B 104 (1981) 315.
[22] D. Friedan, Commun. Math. Phys. 85 (1982) 481.
[23] O. Jahn, J.M. Pawlowski, Nucl. Phys. B 642 (2002) 357;
L. O'Raifeartaigh, Nucl. Phys. B (Proc. Suppl.) 101 (2001) 107.
[24] K. Fujikawa, M. Ishibashi, H. Suzuki, JHEP 0204 (2002) 046;
K. Fujikawa, H. Suzuki, Phys. Rev. D 67 (2003) 034506.
[25] P. Hasenfratz, M. Bissegger, Phys. Lett. B 613 (2005) 57.
[26] H. Nicolai, Nucl. Phys. B 140 (1978) 294;
H. Nicolai, Nucl. Phys. B 156 (1979) 157;
H. Nicolai, Nucl. Phys. B 156 (1979) 177.
[27] P. van Nieuwenhuizen, A. Waldron, Phys. Lett. B 397 (1996) 29.
[28] K. Fujikawa, Nucl. Phys. B 589 (2000) 487.
[29] S. Randjbar-Daemi, J.A. Strathdee, Nucl. Phys. B 466 (1996) 335.
[30] Y. Kikukawa, H. Suzuki, JHEP 0502 (2005) 012.
[31] J. Giedt, E. Poppitz, JHEP 0710 (2007) 076.
[32] J.W. Elliott, J. Giedt, G.D. Moore, Phys. Rev. D 78 (2008) 081701.
[33] E. Poppitz, Y. Shang, arXiv:0901.3402 [hep-lat].
[34] H. Suzuki, JHEP 0010 (2000) 039.
[35] T. Inagaki, H. Suzuki, JHEP 0407 (2004) 038.
[36] P.Y. Huet, R. Narayanan, H. Neuberger, Phys. Lett. B 380 (1996) 291.
[37] P. Hasenfratz, R. von Allmen, JHEP 0802 (2008) 079.
[38] C. Gattringer, M. Pak, Nucl. Phys. B 801 (2008) 353.
[39] I. Montvay, Int. J. Mod. Phys. A 17 (2002) 2377;
I. Montvay, Nucl. Phys. B 466 (1996) 259.
[40] S. Catterall, S. Karamov, Phys. Rev. D 65 (2002) 094501.
[41] M. Beccaria, G.F. De Angelis, M. Campostrini, A. Feo, Phys. Rev. D 70 (2004) 035011;
M. Beccaria, G.F. De Angelis, M. Campostrini, A. Feo, PoS LAT2005 (2006) 269.
[42] T. Kastner, G. Bergner, S. Uhlmann, A. Wipf, C. Wozar, Phys. Rev. D 78 (2008) 095001;
arXiv:0809.2176 [hep-lat].
[43] J. Giedt, R. Brower, S. Catterall, G.T. Fleming, P. Vranas, arXiv:0807.2032 [hep-lat];
J. Giedt, R. Brower, S. Catterall, G.T. Fleming, P. Vranas, Phys. Rev. D 79 (2009) 025015.
[44] I. Kanamori, H. Suzuki, arXiv:0809.2856 [hep-lat];
I. Kanamori, H. Suzuki, arXiv:0811.2851 [hep-lat].
[45] M.G. Endres, arXiv:0902.4267 [hep-lat].